An expression for zeta integer approximation

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1. Introduction

Computing the zeta function for odd integer values is not as easy as for even ones. In this document is exposed a formulae for asymptotic aproximation of these values, using a similar expression to the clasical formulae for odd integers.

2. Exposition

For the zeta function we have the following property

$$\sum_{n=1}^{\infty} \frac{\lambda^2}{n(n-\lambda)} = \sum_{m=2}^{\infty} \zeta(m)\lambda^m; \forall \lambda \in [0,1) \subset \mathbb{R} \tag{1}$$

hence

$$\zeta(m) = \frac{1}{m!} \frac{\partial^m}{\partial \lambda^m} \sum_{n=1}^{\infty} \frac{\lambda^2}{n(n-\lambda)} |_{\lambda=0} \tag{2}$$

Using residues

$$\sum_{m=1}^{\infty} \frac{\lambda^2}{m(m-\lambda)} = \lim_{n\to\infty} \frac{\pi\lambda}{n} \sum_{b=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(\cot\left(\frac{\pi(b-\lambda)}{n} \right) - \cot\left(\frac{\pi b}{n} \right) \right) \tag{3}$$

Let $\phi(n)$ be defined as the right part of (3) expression inside the limit:

$$\phi(n) := \frac{\pi\lambda}{n} \sum_{b=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(\cot\left(\frac{\pi(b-\lambda)}{n} \right) - \cot\left(\frac{\pi b}{n} \right) \right) \tag{4}$$

so that (3) is

$$\sum_{m=1}^{\infty} \frac{\lambda^2}{m(m-\lambda)} = \lim_{n\to\infty} \phi(n)$$

By trigonometric rules

$$\phi(2n) = \phi(n) - \frac{\pi\lambda}{2n} + \frac{\pi\lambda}{2n} \tan\left(\frac{\pi}{4} \left(1 + \frac{2\lambda}{n} \right) \right)) + \frac{\pi\lambda}{2n} \sum_{b=\left\lceil \frac{n+1}{2} \right\rceil}^{n-1} \cot\left(\frac{\pi(b-\lambda)}{2n} \right) + \cot\left(\frac{\pi(b+\lambda)}{2n} \right) - 2\cot\left(\frac{\pi b}{2n} \right)$$

$$\phi(n) = 0 \ \forall n \leq 2$$

Let $w(n) := \phi(2^n)$. Then

$$w(n) = w(n-1) - \frac{\pi\lambda}{2^n} + \frac{\pi\lambda}{2^n} \tan(\frac{\pi}{4}(1 + \frac{\lambda}{2^{n-2}})) + \frac{\pi\lambda}{2^n} \sum_{b=2^{n-2}+1}^{2^{n-1}-1} \cot\left(\frac{\pi(b-\lambda)}{2^n} \right) + \cot\left(\frac{\pi(b+\lambda)}{2^n} \right) - 2\cot\left(\frac{\pi b}{2^n} \right); \forall n > 1$$

$$w(1) = 0$$

In terms of finite differences

$$w(n)=w(n-1)-\frac{\pi\lambda}{2^n}+\frac{\pi\lambda}{2^n}\tan(\frac{\pi}{4}(1+\frac{\lambda}{2^{n-2}}))+\frac{\pi\lambda}{2^n}\sum_{b=2^{n-2}+1}^{2^{n-1}-1}\delta^2_{\frac{\pi\lambda}{2^n}}[\cot](\frac{\pi b}{2^n}) \tag{5}$$

Then

$$w(\infty)=-\frac{\pi\lambda}{2}+\sum_{n=2}^{\infty}\frac{\pi\lambda}{2^n}\tan(\frac{\pi}{4}(1+\frac{\lambda}{2^{n-2}}))+\sum_{m=1}^{\infty}\sum_{b=2^m+1}^{2^{m+1}-1}\frac{\pi\lambda}{2^{m+2}}\delta^2_{\frac{\pi\lambda}{2^{m+2}}}[cot](\frac{\pi b}{2^{m+2}}) \tag{6}$$

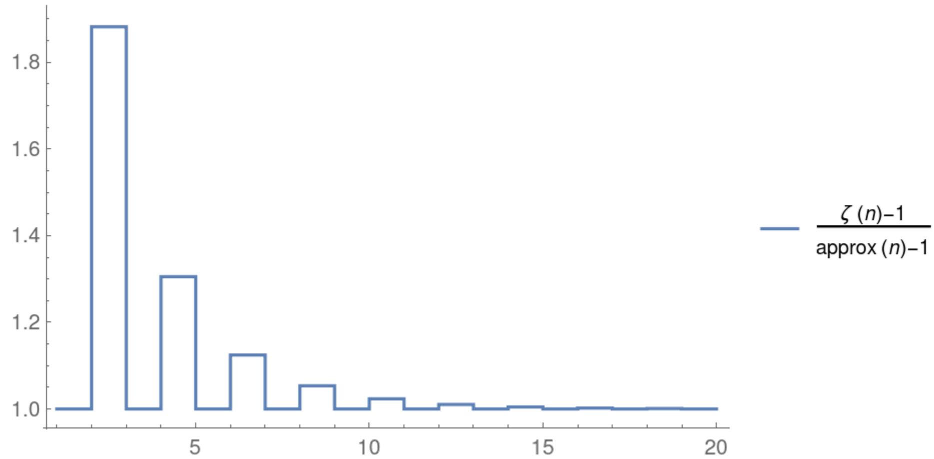
so after simplifying,

$$\zeta(k)=\frac{\pi^k}{4(1-2^{-k})k!}\frac{\partial^k}{\partial\lambda^k}\lambda\tan(\frac{\pi+\lambda}{4})\Big|_{\lambda\to 0}+\frac{\pi^k 2^{-3k}}{(1-2^{-k})k!}\sum_{l=1}^{\infty}2^{-k\lfloor\log_2(l)\rfloor}\frac{\partial^k}{\partial\lambda^k}\lambda\delta^2_{\lambda}[cot](\frac{\pi(2l+1)}{2^{3+\lfloor\log_2(l)\rfloor}})\Big|_{\lambda\to 0}$$

The last right sum vanishes for even k values so the expression

$$\zeta(k)=\frac{\pi^k}{4(1-2^{-k})(k!)}\frac{\partial^k}{\partial\lambda^k}\lambda\tan(\frac{\pi+\lambda}{4})\Big|_{\lambda\to 0} \tag{7}$$

holds for even values, but also for odd values, the right last sum converges fast to zero. The convergence is also good relative to $\lim_{n->\infty}\zeta(n)=1$, as we can see in the following picture.



3. Conclusion

As curiosity, we found an expression that gives us an asymptotic approximations of the zeta function for even bigger integers, always as the product of rational coefficients by powers of pi.

$$\zeta(k)\approx\frac{\pi^k}{4(1-2^{-k})(k!)}\frac{\partial^k}{\partial\lambda^k}\lambda\tan(\frac{\pi+\lambda}{4})\Big|_{\lambda\to 0} \tag{8}$$